J. Fluid Mech. (1971), *vol.* 47, *part* 3, *pp.* 525–535
Printed in Great Britain

Inertial-range transfer in two- and three-dimensional turbulence

By ROBERT H. KRAICHNAN
Dublin, New Hampshire, U.S.A.

(Received 1 July 1970)

A simple dynamical argument suggests that the k^{-3} enstrophy-transfer range in two-dimensional turbulence should be corrected to the form

$$E(k) = C'\beta^{\frac{2}{3}}k^{-3}[\ln(k/k_1)]^{-\frac{1}{3}} \quad (k \gg k_1),$$

where $E(k)$ is the usual energy-spectrum function, β is the rate of enstrophy transfer per unit mass, C' is a dimensionless constant, and k_1 marks the bottom of the range, where enstrophy is pumped in. Transfer in the energy and enstrophy inertial ranges is computed according to an almost-Markovian Galilean-invariant turbulence model. Transfer in the two-dimensional energy inertial range,

$$E(k) = C\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}},$$

is found to be much less local than in three dimensions, with 60 % of the transfer coming from wave-number triads where the smallest wave-number is less than one-fifth the middle wave-number. The turbulence model yields the estimates $C' = 2{\cdot}626$, $C = 6{\cdot}69$ (two dimensions), $C = 1{\cdot}40$ (three dimensions).

1. Introduction

Batchelor (1969), Leith (1968) and Kraichnan (1967) have suggested that two-dimensional turbulence can exhibit two kinds of inertial range: an energy transfer range of the form,

$$E(k) = C\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}, \tag{1.1}$$

and an enstrophy transfer range of the form,

$$E(k) = C'\beta^{\frac{2}{3}}k^{-3}. \tag{1.2}$$

In these equations, $E(k)$ is the usual isotropic energy spectrum, normalized so that

$$\int_0^\infty E(k)\,dk$$

is the average kinetic energy per unit mass, ϵ is the mean rate of kinetic energy transfer per unit mass, β is the mean rate of enstrophy transfer, and enstrophy is defined as half the squared vorticity. C and C' are constants. The wealth of inviscid constants of motion in two-dimensional flow makes the universality of C and C', and, in fact, the very existence of these ranges, much more questionable than for the inertial range in three dimensions. If (1.1) exists, the transfer is from higher to lower wave-numbers, while in (1.2) the enstrophy transfer is from lower to higher wave-numbers. There is zero enstrophy transfer in range (1.1) and zero energy transfer in range (1.2).

Lilly (1969) reports computer experiments which are consistent with (1.1) and (1.2); and he gives rough estimates of C and C', although the Reynolds numbers of the experiments are much too low to give a solid check. Lilly also discusses the possible appearance of (1.2) in atmospheric measurements.

Equation (1.2) implies a logarithmically divergent total enstrophy, which suggests that the -3 law, obtained by assuming local dynamics in k space, should be corrected by a logarithmic-type factor (Kraichnan 1967). The form of the correction can be inferred from a simple argument. We shall start by giving a qualitative dynamical derivation of (1.1). Let $\Pi(k)$ be defined as the total rate of energy transfer from all wave-numbers $< k$ to all wave-numbers $> k$. Kolmogorov's ideas of localness of interaction in k space suggest that $\Pi(k)$ should be proportional to the total energy ($\sim kE(k)$) in wave-numbers $\sim k$ and to some effective rate of shear ω_k which acts to distort flow structures of scale $1/k$. That is,

$$\Pi(k) \sim \omega_k k E(k). \tag{1.3}$$

Furthermore, we expect

$$\omega_k^2 \sim \int_0^k E(p)\, p^2 dp. \tag{1.4}$$

This is because all wave-numbers $\lesssim k$ should contribute to the effective mean-square shear acting on wave-numbers $\sim k$, while the effects of wave-numbers $\gg k$ can plausibly be expected to average out over scales $\sim 1/k$ and over times the order of the characteristic distortion time ω_k^{-1}. Now (1.1) may be inferred by noting that only this choice can give $\Pi(k)$ a value ϵ which is independent of k. Also, we see that, with this choice, the major contribution to (1.4) is from $p \sim k$, in accord with Kolmogorov's localness assumptions. The additional consistency check, that wave-numbers $\gg k$ do not contribute significantly to (1.4), can be made by assuming that random cancellation effects over the domain $1/k$ in linear dimension should reduce the effective shear of the high wave-numbers according to the $\sqrt{N}$ law. These qualitative arguments are corroborated by a computation of $\Pi(k)$ later in this paper according to a closure approximation.

To apply the same kind of argument to conservative enstrophy transfer, we write

$$\Lambda(k) \sim \omega_k k^3 E(k), \tag{1.5}$$

where $\Lambda(k)$ is defined as the total rate of enstrophy transfer from all wave-numbers $< k$ to all wave-numbers $> k$, and $k^3E(k)$ is approximately the enstrophy in wave-numbers $\sim k$. If (1.2) is substituted into (1.4) and (1.5), we would obtain a k-independent value for $\Lambda(k)$ if the major contribution to (1.4) came from $p \sim k$. In fact, however, the integral diverges at the lower limit, so we actually get $\omega_k^2 \propto \ln(k/k_1)$, where k_1 is a wave-number at the bottom of the -3 range, below which the spectrum form changes. This result for ω_k would imply a $\Lambda(k)$ which rises with k, according to (1.5). We may therefore attempt to restore constant $\Lambda(k)$ by trying a correction of the form $E(k) \propto k^{-3}[\ln(k/k_1)]^{-n}$. In this way, we find that

$$E(k) = C'\beta^{\frac{2}{3}} k^{-3}[\ln(k/k_1)]^{-\frac{1}{3}} \quad (k \gg k_1) \tag{1.6}$$

permits $\Lambda(k)$ to have a k-independent value β, with

$$\omega_k^2 \sim \beta^{\frac{2}{3}}[\ln(k/k_1)]^{\frac{2}{3}}. \tag{1.7}$$

If this picture is correct, it shows that the enstrophy transfer range does not exhibit the same degree of localness as the $-\frac{5}{3}$ range. For $k \gg k_1$, most of the contribution to (1.4) comes from $p \gg k_1$, but $p \ll k$. The enstrophy transfer is then analogous to spectral transfer of a passive scalar in the k^{-1} range (Batchelor 1959; Kraichnan 1968), where the dominant straining motion is on a much larger spatial scale than the structures being strained.

It is tempting to try to relate the non-localness of the enstrophy transfer range to a physical picture of the very small scales attached as boundary layers separating larger eddies (Batchelor 1953). This gives a paradox, since then the simple statistical dynamical arguments just stated become very hard to defend.

In the remainder of this paper, we shall verify that (1.1) and (1.6) are consistent with an analytical closure approximation (Kraichnan 1971), thereby providing a more detailed check on the internal consistency of the simple dynamical arguments, although not otherwise making them more persuasive. The closure approximation permits a detailed analysis of the localness of energy and enstrophy transfer in the inertial ranges, and gives an estimate of C and C'. In the case of the $-\frac{5}{3}$ range, we shall compare the energy transfer process with that of a three-dimensional $-\frac{5}{3}$ range.

2. Energy transfer in the $-\frac{5}{3}$ range

In both two and three dimensions, the energy balance equation in stationary isotropic turbulence can be written

$$2\nu k^2 E(k) = T(k) + Z(k), \tag{2.1}$$

where ν is kinematic viscosity, $T(k)$ is the energy transfer function, and $Z(k)$ is the energy input from external stirring. In three dimensions, $Z(k)$ is assumed to be non-zero only for wave-numbers below the inertial range, while in two dimensions we assume $Z(k)$ is confined to the neighbourhood of k_1, from which energy is pumped through the $-\frac{5}{3}$ range, which lies toward smaller k, and enstrophy is pumped through the -3 range, which occupies higher k. $T(k)$ can be written in the form (Kraichnan 1967),

$$T(k) = \frac{1}{2}\int_0^\infty \int_0^\infty dp\,dq\,T(k,p,q), \quad T(k,p,q) = T(k,q,p), \tag{2.2}$$

where $T(k,p,q)\,dp\,dq$ is the net rate of transfer into k from interactions with mode pairs that lie in dp and dq. Then

$$\Pi(k) = \int_k^\infty T(k')\,dk' = \frac{1}{2}\int_k^\infty dk' \int_0^k \int_0^k T(k',p,q)\,dp\,dq$$
$$-\frac{1}{2}\int_0^k dk' \int_k^\infty \int_k^\infty T(k',p,q)\,dp\,dq, \tag{2.3}$$

where the first integral on the right-hand side is the total net input into all wave-numbers $> k$ from interaction with p and q both $< k$, while the second integral is the total net loss to wave-numbers $< k$ from interactions with p and q both $> k$. This exhausts the interactions which can transfer energy across the boundary at k. The limits ∞ for the p and q integrations in (2.3) can be replaced by ∞ and $k'+p$, respectively, since $T(k',p,q)$ is non-zero only if k', p, q can form a triangle.

A $-\frac{5}{3}$ inertial range implies the scaling,

$$T(ak,ap,aq)/T(k,p,q) = a^{-3}, \tag{2.4}$$

if all six wave-numbers are in the range. This permits some transformations which bring (2.3) to the form

$$\Pi(k) = \epsilon = \int_0^1 dv \int_1^{1+v} dw[\ln(w)\, T(w,1,v) + \ln(v)\, T(v,1,w)] \tag{2.5}$$

(Kraichnan 1967). For convenience, we have taken $k = 1$ in (2.5).

Conservation of energy in both two and three dimensions is expressed by

$$T(k,p,q) + T(p,q,k) + T(q,k,p) = 0, \tag{2.6}$$

an identity derivable from the Navier–Stokes equation. In two dimensions, enstrophy conservation gives the additional relation

$$k^2T(k,p,q) + p^2T(p,q,k) + q^2T(q,k,p) = 0, \tag{2.7}$$

which, with (2.6), gives

$$\left.\begin{aligned} T(q,k,p)/T(p,q,k) &= (k^2-p^2)/(q^2-k^2),\\ T(q,k,p)/T(k,p,q) &= (k^2-p^2)/(p^2-q^2).\end{aligned}\right\} \tag{2.8}$$

By using (2.8), (2.5) can be simplified, in two dimensions only, to

$$\Pi(k) = \epsilon = \int_0^1 dv \int_1^{1+v} dw[(1-v^2)\ln(w) + (w^2-1)\ln(v)]\,\frac{T(1,v,w)}{v^2-w^2}. \tag{2.9}$$

Each pair of values v and w in (2.5) or (2.9) corresponds uniquely to a possible *shape* of the triangle formed by the triad of interacting wave-numbers. The integrands therefore display in an interesting fashion the structure of the overall energy transfer. If these equations are written in the form

$$\Pi(k) = \epsilon \int_0^1 Q(v)\,\frac{dv}{v}, \tag{2.10}$$

where $Q(v)$ is defined as v/ϵ times the integral over dw, then $Q(v)$ serves as a measure of the localness of energy transfer. Note that v is the ratio of smallest to middle wave-number in the interacting triad. The function

$$W(v) = \int_v^1 Q(s)\,\frac{ds}{s} \tag{2.11}$$

is the fraction of the total inertial-range transfer that is due to all triad interactions in which the ratio of smallest to middle wave-number is not less than v.

A number of closure approximations for isotropic turbulence proposed in the last fifteen years lead to the following form for $T(k,p,q)$ in three dimensions:

$$\begin{aligned} T(k,p,q) \\ = 4\pi^2k^3pq[2a_{kpq}\theta_{kpq}U(p)\,U(q) - b_{kpq}\theta_{pqk}U(k)\,U(q) - b_{kqp}\theta_{qkp}U(k)\,U(p)],\end{aligned} \tag{2.12}$$

where $U(k) = (2\pi k^2)^{-1} E(k)$ is the mode intensity,

$$b_{kpq} = pk^{-1}(xy+z^3), \quad 2a_{kpq} = b_{kpq} + b_{kqp}, \tag{2.13}$$

and x, y, z are the cosines of the interior angles opposite k, p, q in the triangle formed by the latter. $T(k,p,q)$ vanishes if k, p, q cannot form a triangle. The corresponding form in two dimensions is

$$T(k,p,q) = 2\pi k^3(1-x^2)^{-\frac{1}{2}}[2\bar{a}_{kpq}\theta_{kpq}U(p)\,U(q) - \bar{b}_{kpq}\theta_{pqk}U(k)\,U(q) - \bar{b}_{kqp}\theta_{qkp}U(k)\,U(p)], \tag{2.14}$$

where $U(k) = (\pi k)^{-1} E(k)$

and
$$\bar{b}_{kpq} = 2pk^{-1}(xy - z + 2z^3), \quad 2\bar{a}_{kqp} = \bar{b}_{kqp} + \bar{b}_{kqp}. \tag{2.15}$$

The quantity θ_{kpq} has the dimensions of time. It is an effective interaction or memory time for the triad k, p, q, and its form depends on which approximation is used. First of all, $\theta_{kpq} = t$ in (2.12) and (2.14) gives the asymptotically exact $T(k,p,q)$ for very small t, if the different wave-numbers are statistically independent at $t = 0$.† The single-time quasi-normal approximation (Proudman & Reid 1954; Tatsumi 1957), Edwards's (1964) theory, the direct-interaction approximation (Kraichnan 1964), Herring's (1965, 1966) theory, the Lagrangian-history direct-interaction approximation (Kraichnan 1966), and an almost-Markovian turbulence model (Kraichnan 1971) that will be used in this paper all yield for θ_{kpq} the steady-state form,

$$\theta_{kpq} = \int_0^\infty G_k(t)\,r_p(t)\,r_q(t)\,dt, \tag{2.16}$$

where $G_k(t)$ and $r_k(t)$ are characteristic response and correlation functions for wave-number k whose particular choice makes the difference between the approximations. Of the approximations named, the Lagrangian-history direct-interaction approximation and the almost-Markovian model yield $-\frac{5}{3}$ inertial ranges.

We wish to demonstrate now that the structure of the inertial-range energy transfer, as exhibited by $Q(v)$, is remarkably insensitive to the precise form of θ_{kpq}, a fact which adds to whatever confidence (2.12) and (2.14) may inspire. Figure 1 shows the results for $Q(v)$ obtained by substituting (1.1) into (2.12) and (2.14) and taking the two parametrizations,

$$\theta_{kpq} = (\eta_k + \eta_p + \eta_q)^{-1}, \tag{2.17a}$$

$$\theta_{kpq} = (\eta_k^2 + \eta_p^2 + \eta_q^2)^{-\frac{1}{2}}, \tag{2.17b}$$

with
$$\eta_k = C^{\frac{1}{2}}\mu\epsilon^{\frac{1}{3}}k^{\frac{2}{3}}. \tag{2.18}$$

These two forms correspond to $G_k(t) = r_k(t)$ and to exponential and Gaussian forms, respectively, for $G_k(t)$. Equation (2.18), with μ a dimensionless proportionality constant, and $C^{\frac{1}{2}}$ present for later convenience, is the form required for there to be a $-\frac{5}{3}$ range. A third choice,

$$\theta_{kpq} = (2\eta_k + \eta_p + \eta_q)^{-1},$$

† The expression for $T(k, p, q)$ given by Kraichnan (1967) is wrong by a factor π.

which corresponds to $G_k(t) = r_k(2t)$ and exponential $G_k(t)$, yields curves for $Q(v)$ which lie between those plotted, both in three dimensions and in two dimensions. The $Q(v)$ curve for the abridged Lagrangian-history direct-interaction approximation (Kraichnan 1966), which has been evaluated only for three dimensions, lies comparably close. The Galilean-invariant almost-Markovian model (Kraichnan 1971), which we shall use to estimate C and C', gives precisely (2.17a).

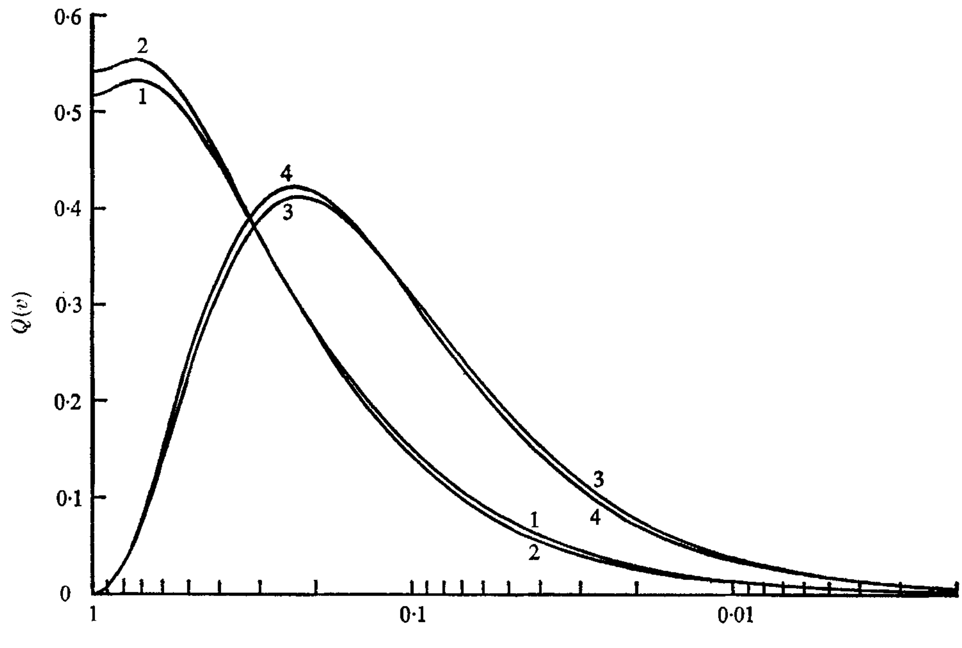

FIGURE 1. Localness of energy transfer. Curves 1 and 2: three dimensions, with (2.17a) and (2.17b), respectively. Curves 3 and 4: two dimensions, with (2.17a) and (2.17b), respectively.

Figure 2 shows the curves for $W(v)$ obtained in three and two dimensions, using (2.17a). Together with figure 1, it suggests a substantial difference in the structure of the energy transfer in the two geometries. In three dimensions, the transfer is already not very local; 65 % of the transfer involves wave-number triads in which the smallest wave-number is less than one-half of the middle wave-number. In two dimensions, 60 % of the transfer involves triads where this ratio is less than one-fifth. The extreme diffuseness of transfer in two dimensions suggests that a very extensive inertial range may be needed to give a really close approach to asymptotic spectrum levels.

In both two and three dimensions, we find (from either (2.17a) or (2.17b))

$$Q(v) = O[v^{\frac{4}{3}} \ln (1/v)] \quad (v \ll 1). \tag{2.19}$$

This corroborates the localness of interaction needed for the qualitative argument of § 1.

The calculations of $\Pi(k)$ with (2.17) give $\Pi(k) > 0$ in three dimensions, and

$\Pi(k) < 0$ in two dimensions, again corroborating qualitative arguments. The vanishing of $Q(1)$ in two dimensions, but not in three, is because, in the former case, each triad interaction conserves both energy and enstrophy, a property that is preserved in (2.14). This implies that $T(k,p,q)$ is zero if any two of the three wave-numbers are equal, as can be seen from (2.6) and (2.7).

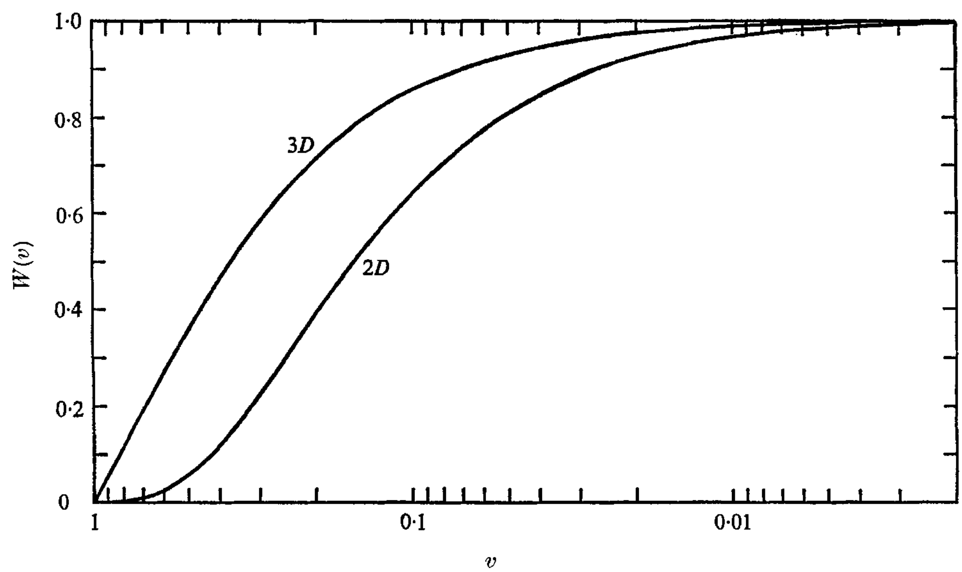

FIGURE 2. The function $W(v)$ in three dimensions (3D) and two dimensions (2D).

3. Estimation of C

With the quadratures carried out, (1.1), (2.5), (2.12), (2.14) and (2.17a) give

$$C = 3{\cdot}022\mu^{\frac{2}{3}} \text{ (three dimensions)}, \quad C = 8{\cdot}94\mu^{\frac{2}{3}} \text{ (two dimensions)}. \tag{3.1}$$

To evaluate μ, we shall use the almost-Markovian Galilean-invariant turbulence model (Kraichnan 1971), in which the memory times θ_{kpq} are fixed by the interaction between the solenoidal and compressive parts of an advected test field. For convenience, we shall call this the 'test-field model'. In the three-dimensional steady state, the test-field model determines η_k by the coupled equations,

$$\eta_k = \nu k^2 + \pi k g^2 \iint_\Delta b^G_{kpq} U(q)\,(\eta_k + \eta^C_p + \eta_q)^{-1} pq\,dp\,dq, \tag{3.2}$$

$$\eta^C_k = \nu k^2 + 2\pi k g^2 \iint_\Delta b^G_{kpq} U(q)\,(\eta^C_k + \eta_p + \eta_q)^{-1} pq\,dp\,dq, \tag{3.3}$$

where
$$b^G_{kpq} = \tfrac{1}{2}(1-y^2)(1-z^2), \tag{3.4}$$

and $\iint_\Delta$ denotes integration over all of the p, q plane where k, p, q can form a triangle. The quantities η_k and η^C_k arise in the model as the relaxation frequencies of the solenoidal and compressive parts of the test field, respectively

$$(\eta_k = \nu k^2 + \eta^S(k,t), \quad \eta^C_k = \nu k^2 + \eta^C(k,t)$$

in the notation of Kraichnan 1971). The quantity g is a scaling constant. It was taken as

$$g = 1{\cdot}064, \tag{3.5}$$

in order to make the test-field model reproduce results of the direct-interaction approximation for the relaxation of small departures from absolute statistical equilibrium in the interaction of the modes in a thin spherical shell, this being a case where the direct-interaction approximation is expected to be accurate.

In the inertial range, we neglect the νk^2 terms, assume the forms (1.1), (2.18), and

$$\eta_k^C = C^{\frac{1}{2}}\mu^C\epsilon^{\frac{1}{3}}k^{\frac{2}{3}}, \tag{3.6}$$

and solve (3.2), (3.3) by iteration to find

$$\mu = 0{\cdot}296g, \quad \mu^C/\mu = 2{\cdot}163 \quad \text{(three dimensions)}. \tag{3.7}$$

In two dimensions, the test-field model gives $\eta_k^C = \eta_k$ and

$$\eta_k = \nu k^2 + 4k^2g^2 \iint_\Delta (1-x^2)^{-\frac{1}{2}}\, b_{kpq}^G\, U(q)\,(\eta_k+\eta_p+\eta_q)^{-1}\,dp\,dq. \tag{3.8}$$

Use of (1.1) and (2.18) then gives

$$\mu = 0{\cdot}609g, \quad \mu^C/\mu = 1 \quad \text{(two dimensions)}. \tag{3.9}$$

Putting these numerical values together, we find

$$C_{3D} = 1{\cdot}40, \quad C_{2D} = 6{\cdot}69, \quad C_{2D}/C_{3D} = 4{\cdot}78. \tag{3.10}$$

Note that C_{2D}/C_{3D}, the ratio of C values for two and three dimensions, is independent of g.

The convergence of the integrals in (3.2), (3.3) and (3.8), when the inertial-range forms are substituted, is a corroboration, in the context of the test-field model, of the localness argument for ω_k, given in § 1.

4. Estimation of C'

The enstrophy transfer function is $k^2T(k)$ and, in analogy to $\Pi(k)$, the net rate at which enstrophy is transferred from all wave-numbers $< k$ to all wave-numbers $> k$ is

$$\Lambda(k) = \frac{1}{2}\int_k^\infty dk' \int_0^\infty\int_0^\infty (k')^2\,T(k',p,q)\,dp\,dq. \tag{4.1}$$

Suppose $q' \ll k$. Then the total contribution to $\Lambda(k)$ from all triad interactions involving a wave-number $\leqslant q'$ is, in correspondence to (2.3),

$$\Lambda_{<q'}(k) = \int_k^\infty dk' \int_0^k dp \int_0^{q'} dq(k')^2\,T(k',p,q) - \frac{1}{2}\int_0^{q'} dk' \int_k^\infty\int_k^\infty dp\,dq(k')^2\,T(k',p,q), \tag{4.2}$$

where there is no factor $\frac{1}{2}$ in the first integral because either p or q in (4.1) can be $< q'$. Equation (4.2) is readily evaluated if we take (2.14) and (2.17a). The second term on the right-hand side turns out to be negligible compared to the first, as a consequence of (2.8) and the fact that T vanishes for an equilateral triangle. This means the physically reasonable result that a negligible fraction of the net enstrophy transfer is into the wave-numbers $< q'$. To evaluate the first term, we use the triangle condition to rewrite the integration limits as

$$\int_k^{k+q} dk' \int_{k'-q}^{k} dp \int_0^{q'} dq.$$

Then we assume $\eta_{k'} \approx \eta_p \approx \eta_k \gg \eta_q$, expand the various factors in the integrand in powers of q/k', and, to leading order in q'/k, find

$$\Lambda_{<q'}(k) = -(\tfrac{1}{16}\pi^2)\,k^3(\eta_k)^{-1}\frac{d\Omega(k)}{dk}\int_0^{q'} U(q)\,q^3 dq \quad (q' \ll k), \tag{4.3}$$

where $\Omega(k) = k^2 U(k)$ is the enstrophy mode intensity.

The integral in (4.3) is proportional to the mean-square shear in wave-numbers $< q'$, and the right-hand side vanishes if $\Omega(k)$ is independent of k, corresponding to equi-partition of enstrophy. In these respects, (4.3) resembles a similar expression for transfer of a passive scalar in the k^{-1} spectrum range according to the Lagrangian-history direct-interaction approximation (Kraichnan 1968).

Now suppose that $E(k)$ has the form (1.6), and that k/k_1 is sufficiently large that we can find a q' such that $k_1 \ll q' \ll k$ and $\ln(k/q') \ll \ln(k/k_1)$. If this $E(k)$ is substituted into (4.3), we find, to leading order,

$$\Lambda_{<q'}(k) = \tfrac{3}{16}(C')^2\beta^{\frac{4}{3}}(\eta_k)^{-1}[\ln(k/k_1)]^{\frac{1}{3}}. \tag{4.4}$$

On the other hand, if we substitute (1.6) into (3.8), neglect νk^2, and assume $\eta_k \approx \eta_p \gg \eta_q$, we can easily carry out the needed quadrature and find that, to leading order in q'/k, the contribution to η_k from all $q < q'$ is

$$[\eta_k]_{<q'} = \tfrac{9}{16}g^2 C'\beta^{\frac{2}{3}}(\eta_k)^{-1}[\ln(k/k_1)]^{\frac{2}{3}}, \tag{4.5}$$

where we put the same conditions on q' as in (4.4).

It is easily verified that (4.4) and (4.5) are the dominant contributions to $\Lambda(k)$ and η_k in the range (1.6), as expected from the arguments given in § 1. With (4.5) assumed as the asymptotic form of η_k, wave-numbers in the range q' to k give corrections involving $\ln(k/q')$, while very large wave-numbers give converging contributions to the integrals for $\Lambda(k)$ and η_k because of the triangle restriction on interacting triads. We have, finally, then, the asymptotic results,

$$\Lambda(k) = \tfrac{3}{16}(C')^2\beta^{\frac{4}{3}}(\eta_k)^{-1}[\ln(k/k_1)]^{\frac{1}{3}} \quad (k \gg k_1), \tag{4.6}$$

$$\eta_k = \tfrac{3}{4}g(C')^{\frac{1}{2}}\beta^{\frac{1}{3}}[\ln(k/k_1)]^{\frac{1}{3}} \quad (k \gg k_1). \tag{4.7}$$

Substituting (4.7) into (4.6) and requiring $\Lambda(k) = \beta$, we find

$$C' = (4g)^{\frac{2}{3}} = 2{\cdot}626, \quad C'/C_{2D} = 0{\cdot}393, \tag{4.8}$$

the last result being independent of g.

5. Experiments

The principal question about the ranges (1.1) and (1.6) is whether they are valid asymptotic representations of possible turbulent flows and, if so, how nearly universal the constants C and C' may be. Simple closure approximations, like the one used in this paper, can verify only the self-consistency of the ranges on a rather crude level; they leave untouched the question of existence. There are, nevertheless, some strong hints in the theoretical results. First, the energy and enstrophy transfer formulas indicate a highly non-local interaction in k space, much more non-local in two dimensions than in three, in the case of the $-\frac{5}{3}$ range. We have seen that this result is surprisingly insensitive to the precise way in which the memory times in the closure equations are constructed, the latter being the most arbitrary feature of the closure approximation. This non-localness is consistent with a picture of eddy structures in two-dimensional turbulence having rather high organization, and it suggests that much higher Reynolds numbers may be needed in two-dimensional than in three-dimensional flow in order to test cleanly for the asymptotic ranges. The second major hint from the closure approximation is that C_{2D}/C_{3D} is substantially larger than one. However, a corollary of the diffuseness of transfer is that C_{2D} may appear much less universal, in any practicable experiment or computer experiment, than C_{3D}.

In order to exhibit the logarithmic correction in (1.6), extremely large wave-number ratios would be needed, so that this would appear quite safe from experimental confrontation.

The computer experiments reported by Lilly (1969) appear, at face value, to be consistent with the existence of $-\frac{5}{3}$ and -3 ranges in two-dimensional turbulence, and to suggest values of C and C' reasonably close to those obtained from the closure. However, it is clear from the statements just made that there is no theoretical reason, as yet, to expect the asymptotic ranges to exhibit themselves at the low Reynolds numbers accessible to the computer experiments. The other possible hunting ground for the inertial ranges, atmospheric data, poses the difficulty of separating off other dynamical processes.

The most reasonable kind of experimental test would appear to be running computer experiments and integrations of the closure approximation for a variety of initial conditions, forcing functions, and Reynolds numbers within the range of the experiments, and thereby to see how well the quantitative accuracy of the closure approximation holds up with increasing Reynolds number. This might suggest how much confidence could be placed in using the closure results at the still higher Reynolds numbers which are inaccessible to controlled experiment.

The value $C = 1{\cdot}40$, that we have found for the $-\frac{5}{3}$ range in three dimensions, is close to the value 1·44 reported by Grant, Stewart & Moilliet (1962). However, a best fit to the experiment appears to give a value for C at least 10 % higher than the reported value 1·44, if account is taken of dissipation-range effects. The abridged Lagrangian-history direct-interaction inertial and dissipation-range spectrum (Kraichnan 1966) gives an excellent fit to the data, and yields $C = 1{\cdot}77$.

I am grateful to Dr D. K. Lilly for discussions, and for making results of computer experiments available before publication. This research was supported by the Atmospheric Sciences Section, National Science Foundation, N.S.F. Grant GA-16186.

REFERENCES

BATCHELOR, G. K. 1953 *Theory of Homogeneous Turbulence.* Cambridge University Press.
BATCHELOR, G. K. 1959 *J. Fluid Mech.* **5**, 113.
BATCHELOR, G. K. 1969 *Phys. Fluids,* **12**, II-233.
EDWARDS, S. F. 1964 *J. Fluid Mech.* **18**, 239.
GRANT, H. L., STEWART, R. W. & MOILLIET, A. 1962 *J. Fluid Mech.* **12**, 241.
HERRING, J. 1965 *Phys. Fluids,* **8**, 2219.
HERRING, J. 1966 *Phys. Fluids,* **9**, 2106.
KRAICHNAN, R. H. 1964 *Phys. Fluids,* **7**, 1163.
KRAICHNAN, R. H. 1966 *Phys. Fluids,* **9**, 1728.
KRAICHNAN, R. H. 1967 *Phys. Fluids,* **10**, 1417.
KRAICHNAN, R. H. 1968 *Phys. Fluids,* **11**, 945.
KRAICHNAN, R. H. 1971 *J. Fluid Mech.* **47**, 513.
LEITH, C. E. *Phys. Fluids,* **11**, 671.
LILLY, D. K. 1969 *Phys. Fluids,* **12**, II–240.
PROUDMAN, I. & REID, W. H. 1954 *Phil. Trans.* A, **247**, 163.
TATSUMI, T. 1957 *Proc. Roy. Soc.* A **239**, 16.